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Intuitionistic fuzzy theory and its application in economy, technology and management

AN APPROACH FOR MADM PROBLEMS WITH INTERVAL-VALUED INTUITIONISTIC FUZZY SETS BASED ON NONLINEAR FUNCTIONS

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Abstract. This paper investigates an approach for multiple attribute decision making (MADM) problems with interval-valued intuitionistic fuzzy numbers (IVIFNs). To do that, the nonlinear score, accuracy and hesitation functions of IVIFNs are developed based on the normal distribution. The novelty of these nonlinear functions is that they have an additional variance value, which can have more information to rank IVIFNs than Xu and Chen's score function and Ye's accuracy function. Based on these nonlinear functions, a ranking method for IVIFNs is proposed. Furthermore, a nonlinearly optimized model is proposed to obtain attribute weights by integrating these nonlinear functions. Then, we develop an approach for interval-valued intuitionistic fuzzy MADM programs in which two cases are considered: the attribute weight information is known and particularly known. In the end, we apply the proposed approach to select green supplier.

Keywords: multi-attribute decision-making, interval-valued intuitionistic fuzzy set, score function, accuracy function, hesitation function, normal distribution.

JEL Classification: C021, C441, D81, N55.

Introduction

The theory of fuzzy sets (FSs) proposed by Zadeh (1965) is a powerful tool to deal with vagueness, whose basic component is only a membership function. Atanassov (1986) introduced the concept of intuitionistic fuzzy set (IFS) characterized by a membership function and a non-membership function, which is an extension of Zadeh's fuzzy sets. Later, Atanassov and Gargov (1989) introduced the concept of interval-valued intuitionistic fuzzy sets (IVIFSs) as a further generalization of that of IFSs. The intuitionistic and interval-valued intuitionistic fuzzy set theory has been applied to many different fields, such as multiplicative criteria

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decision making with complete weight information of attribute (Zhang, Liu 2010; Jiang *et al.* 2011; Park *et al.* 2011; Wei 2011; Yang, Chiclana 2009, 2012), multiplicative attribute decision making with incomplete weight information of attribute (Wang *et al.* 2009a, 2011a, 2011b; Pei, Zheng 2012; Yue 2011a; Zhao *et al.* 2012), multiplicative attribute decision making with intuitionistic or interval-valued intuitionistic fuzzy preference relations (Gong *et al.* 2009, 2011; Wu, Chiclana 2012; Xu 2013), group decision making (Li 2007; Li *et al.* 2010; Chen, Yang 2011; Su *et al.* 2011; Yue 2011b; Wei *et al.* 2012; Xu 2012; Zeng 2013; Xia, Xu 2013), aggregating operators (Wei 2009, 2010; Li 2011; Liu 2011; Merigó 2011; Merigó, Gil-Lafuente 2011;Wu, Cao 2013; Wu 2015), supplier selection (Boran *et al.* 2009), virtual enterprise partner selection (Ye 2010), strategy selection (Wei, Merigó 2012). In the decision making with intuitionistic or interval-valued intuitionistic fuzzy numbers, one key issue that needs to be addressed is to rank IFNs and IVFNs. Xu and Chen (2007) defined the score function and accuracy function to IVIFNs environments, and then developed an approach to rank IVFNs. However, in some cases, these functions do not allow the proper discrimination between different IVFNs. To resolve this problem, this article aims to develop the nonlinear score, accuracy and hesitation functions of IVIFNs based on the normal distribution, and then investigate a novel ranking approach.

The ranking problem has been extensively studied for the case of fuzzy numbers (FNs). A widely used approach to rank FNs is to convert them into a representative crisp value, and perform the comparison on them (Yager 2004), which is also the common one used to rank IFNs and IVIFNs. Representative crisp values developed for IFNs and IVIFNs are known with the names of score degree and accuracy degree. By the membership and non-membership functions, Chen and Tan (1994) developed a score function for IFSs, which was later improved by Hong and Choi (2000) with the addition of an accuracy function. Other score and accuracy functions had been proposed in (Li *et al.* 2007; Wang *et al.* 2009b). Xu and Chen (2007) extended the score function and accuracy function to IVIFNs environments. Later, Ye (2009) proposed a different accuracy function that he claimed solved some drawbacks associated to the accuracy function developed by Xu and Chen (2007). However, both accuracy functions were proved to be equivalent when ordering IVIFNs (Wang 2011), and therefore the drawbacks highlighted by Ye (2009) were not properly addressed. Wu and Chiclana (2014) investigated a risk attitudinal score and accuracy expected functions to rank IVIFNs. Recently, Lakshmana Gomathi Nayagam and Sivaraman (2012) claimed that a novel accuracy function is proposed to compare IVIFNs. However, this accuracy function can be proved to be a score function more than an accuracy function. Indeed, in some cases, these proposals do not allow the proper discrimination between different IVIFNs. One key reason is that there is no research on the rationality of these functions, which may lead to misuse them in the process of raking IVIFNs. Another key reason is that they are straight forward extensions of their respective proposals for the case of IFNs. However, IVIFNs are more complicate than IFNs because that their membership and non-membership functions are interval numbers, which are nonlinear functions and can not be compared directly. Therefore they are not rich enough to capture all the information contained in IVIFNs.

To resolve these problems, we firstly build some judgment criterions to study the rationality of score and accuracy functions. Secondly, we develop the nonlinear score, accuracy and hesitation functions of IVIFNs based on the normal distribution. Our nonlinear functions extends: (i) Xu and Chen's score function for IVIFNs.; and (ii) Ye's accuracy function for IVIFNs. Then, we give an order relation between IVIFNs. This method guarantees that it can give sufficient information about interval-valued intuitionistic fuzzy numbers based on the score values, variance values and accuracy values. Finally, an approach for MADM programs with IVIFNs is proposed in which two cases are considered: the attribute weight information is known and particularly known.

The rest of this paper is organized as follows. In Section 1, we introduce some basic concepts related to IVIFSs. Some propositions are proposed to study the rationality of score and accuracy functions. Section 2 proposes the normal distribution based score accuracy and hesitation functions. In Section 3, an optimization model is developed to determine the attribute weights based on the nonlinear functions. Section 4 proposes an approach for MADM programs with IVIFNs in which two cases are considered: the attribute weight information is known and particularly known. Section 5 gives an illustrative numerical example to verify the developed approach. Finally, in the last Section we draw our conclusions.

1. Preliminaries

We start this section by introducing some basic concepts related to interval-valued intuitionistic fuzzy sets, which will be used throughout this paper.

Atanassov and Gargov (1989) introduced the notion of interval-valued intuitionistic fuzzy set (IVIFS), which is characterized by a membership function and a non-membership function, whose values are intervals rather than exact numbers.

Definition 1 *(IVIFS of Atanassov and Gargov (1989))***.** Let $D \in [0,1]$ be the set of all closed subintervals of the interval and X be a universe of discourse. An interval-valued intuitionistic fuzzy set in A over X is an object having the form:

$$A = \{< x, \tilde{\mu}_A(x), \tilde{\nu}_A(x) > \big| x \in X\}$$

where

$$\tilde{\mu}_A \to D \in [0,1], \quad \tilde{\nu}_A(x) \to D \in [0,1]$$

with the condition $0 \le \sup \tilde{\mu}_A(x) + \sup \tilde{\nu}_A(x) \le 1, \forall x \in X$.

The intervals $\tilde{\mu}_A(x)$ and $\tilde{\nu}_A(x)$ denote, respectively, the membership function and the non-membership function of the element x to the set A. Thus for each $x \in X$, $\tilde{\mu}_A(x)$ and $\tilde{\nu}_A(x)$ are closed intervals and their lower and upper end points are, respectively, denoted by $\tilde{\mu}_{AL}(x)$, $\tilde{\mu}_{AU}(x)$, $\tilde{\nu}_{AL}(x)$ and $\tilde{\nu}_{AU}(x)$. We can denote by

$$A = \{< x, [\tilde{\mu}_{AL}(x), \tilde{\mu}_{AU}(x)], [\tilde{\nu}_{AL}(x), \tilde{\nu}_{AU}(x)] > \big| x \in X\},$$

where

$$0 \le \tilde{\mu}_{AU}(x) +, \tilde{\nu}_{AU}(x) \le 1, \; \tilde{\mu}_{AL}(x) \ge 0, \tilde{\nu}_{AL}(x) \ge 0.$$

For each element *x*, we can compute the unknown degree (hesitancy degree) of an interval-valued intuitionistic fuzzy interval of $x \in X$ in $\tilde{A}$ defined as follows:

$$\pi_{A(x)} = 1 - \tilde{\mu}_{\tilde{A}}(x) - \tilde{\nu}_{\tilde{A}}(x) = [1 - \tilde{\mu}_{AU}(x) - \tilde{\nu}_{AU}(x)], [1 - \tilde{\mu}_{AL}(x) - \tilde{\nu}_{AL}(x)].$$

We will denote the set of all the IVIFSs in X by IVIFS(X). For convenience, let $\tilde{\mu}_A(x) = [a,b]$, $\tilde{\nu}_A(x) = [c,d]$, so $A = ([a,b],[c,d])$.

Xu and Chen (2007) proposed the following score and accuracy functions associated to an IVIFN:

Definition 2 *(Score function of Xu and Chen (2007))*. Let $A = ([a,b],[c,d])$ be an interval-valued intuitionistic fuzzy number, a score function *S* of an interval-valued intuitionistic fuzzy value can be represented as follows:

$$S(A) = \frac{a+b-c-d}{2}, \; S(A) \in [-1,1], \tag{1}$$

to evaluate the degree of score of the interval-valued intuitionistic fuzzy value $A = ([a,b],[c,d])$, where $S(A) \in [-1,1]$. The larger the value of $S(A)$, the more the degree of score of the interval-valued intuitionistic fuzzy value *A*.

Definition 3 (Accuracy function of Xu and Chen (2007)). Let $A = ([a,b],[c,d])$ be an interval-valued intuitionistic fuzzy number, an accuracy function *H* of an interval-valued intuitionistic fuzzy value can be represented as follows:

$$H(A) = \frac{a+b+c+d}{2}, \; H(A) \in [0,1], \tag{2}$$

to evaluate the degree of accuracy of the interval-valued intuitionistic fuzzy value $A = ([a,b],[c,d])$, where $H(A) \in [0,1]$. The larger the value of $H(A)$, the more the degree of accuracy of the interval-valued intuitionistic fuzzy value *A*.

Ye (2009) proposed a different expression for the accuracy degree of an IVIFN that has the same range of values $[-1, 1]$, than the score degree defined above:

Definition 4 (*Accuracy function of Ye* (2009))**.** Let $A = ([a,b],[c,d])$ be an interval-valued intuitionistic fuzzy number, a novel accuracy function *M* of an interval-valued intuitionistic fuzzy value can be represented as follows:

$$M(A) = a + b - 1 + \frac{c+d}{2}, \tag{3}$$

where $M(A) \in [-1,1]$. The larger the value of $M(A)$, the more the degree of accuracy of the interval-valued intuitionistic fuzzy value A.

Lakshmana Gomathi Nayagam and Sivaraman (2012) developed a new accuracy function of IVIFSs as follows:

Definition 5 *(Accuracy function of Lakshmana Gomathi Nayagam and Sivaraman (2012))*. Let $A = \left([a,b],[c,d]\right)$ be an interval-valued intuitionistic fuzzy number, a novel accuracy function L of an interval-valued intuitionistic fuzzy value can be represented as follows:

$$L(A) = \frac{a+b-d(1-b)-c(1-a)}{2}. \tag{4}$$

The larger the value of $L(A)$, the more the degree of accuracy of the interval-valued intuitionistic fuzzy value A.

These functions have been universally used in decision making problems with IVIFNs. However, in some cases, these proposals do not allow the proper discrimination between different IVIFNs.

Example 1. If interval-valued intuitionistic fuzzy values for two alternatives are $\tilde{\alpha}_1 = ([0.05,0.35],[0.2,0.5])$ and $\tilde{\alpha}_2 = ([0.1,0.3],[0.3,0.4])$, then the desirable alternative is selected in accordance with accuracy function.

By applying Definition 2 and Definition 3, we can obtain $S(\tilde{a}_1) = S(\tilde{a}_2) = -0.15$ and $H(\tilde{a}_1) = H(\tilde{a}_2) = 0.55$, respectively. In this case we do not know which alternative is better. By applying Definition 4, we have $M(\tilde{a}_1) = -0.25$ and $M(\tilde{a}_2) = -0.25$. In both the cases we do not know which alternative is better. But, according to Definition 5, we get $L(\tilde{a}_1) = 0.015$ and $L(\tilde{a}_2) = -0.08$, and hence $\tilde{a}_1$ is better than $\tilde{a}_2$.

Example 2. If interval-valued intuitionistic fuzzy values for two alternatives are $\tilde{\alpha}_1 = ([0.2,0.2],[0.3,0.5])$ and $\tilde{\alpha}_2 = ([0.2,0.2],[0.1,0.7])$, then the desirable alternative is selected in accordance with accuracy function.

According to Definition 2 and Definition 3, we can obtain $S(\tilde{a}_1) = S(\tilde{a}_2) = -0.2$ and $H(\tilde{a}_1) = H(\tilde{a}_2) = 0.6$, respectively. In this case we do not know which alternative is better. According to Definition 4, we have $M(\tilde{a}_1) = M(\tilde{a}_2) = -0.2$. By Definition 5, we get $L(\tilde{a}_1) = -0.12$ and $L(\tilde{a}_2) = -0.12$. In these three cases, we do not know which alternative is better.

The above examples demonstrate the limitation of these functions. However, the reasons for this limitation have not been discovered and discussed. To study the properties of score function and accuracy function, we propose the following propositions.

Proposition 1 *(Monotonicity of score function)*. Let $A = \left([a,b],[c,d]\right)$ be an interval-valued intuitionistic fuzzy number, the score function $S(A) = (a+b-c-d)/2$ is a monotone increasing function with a and b, and a monotone decreasing function with c and d.

Proof. Omited.

Proposition 2 *(Symmetry of score function)*. Let $A_1 = \left([a_1,b_1],[c_1,d_1]\right)$ and $A_2 = \left([a_2,b_2],[c_2,d_2]\right)$ be two IVIFNs, $\overline{A}_1 = \left([c_1,d_1],[a_1,b_1]\right)$ and $\overline{A}_2 = \left([c_2,d_2],[a_2,b_2]\right)$ be their associated inverse functions, receptively, then we have the following conclusion $S(A_1) \le S(A_2) \Leftrightarrow S(\overline{A}_1) \ge S(\overline{A}_2)$.

Proof. (Sufficiency) From Definition 2, we obtain:

$$S(A_1) = \frac{a_1+b_1-c_1-d_1}{2} \text{ and } S(A_2) = \frac{a_2+b_2-c_2-d_2}{2}.$$

Since $S(A_1) \le S(A_2)$, then

$$a_1 + b_1 - c_1 - d_1 \le a_2 + b_2 - c_2 - d_2 .$$

That is:

$$c_1 + d_1 - a_1 - b_1 \le c_2 + d_2 - a_2 - b_2 .$$

Thus, we have:

$$S(\overline{A}_1) = \frac{c_1 + d_1 - a_1 - b_1}{2} \le \frac{c_2 + d_2 - a_2 - b_2}{2} = S(\overline{A}_2) .$$

We also can get the proof of necessity. The proof of Proposition 2 is completed.

Proposition 3 *(Monotonicity of accuracy function)***.** Let $A = ([a,b],[c,d])$ be an interval-valued intuitionistic fuzzy number, the accuracy functions $H(A)$, $M(A)$ and $L(A)$ are the monotone increasing functions with a, b, c and d.

Proof. Omited.

Proposition 4 *(Symmetry of accuracy function)***.** Let $A_1 = ([a_1,b_1],[c_1,d_1])$ be an IVIFN and $\overline{A}_1 = ([c_1,d_1],[a_1,b_1])$ be its associated inverse function, then we have the following conclusions 1) $H(A_1)=H(\overline{A}_1)$, 2) $M(A_1) \ne M(\overline{A}_1)$, and 3) $L(A_1) \ne L(\overline{A}_1)$.

Proof. 1) According the Definition 3, we have:

$$H(A_1)=\frac{a_1 + b_1 + c_1 + d_1}{2}=H(\overline{A}_1) .$$

2) According the Definition 4, we obtain:

$$M(A_1)=a_1 + b_1 + \frac{c_1 + d_1}{2} - 1 ,$$

and

$$M(\overline{A}_1)=c_1 + d_1 + \frac{a_1 + b_1}{2} - 1 ,$$

then,

$$M(A_1) \ne M(\overline{A}_1).$$

3) According the Definition 5, we obtain:

$$L(A_1) = \frac{a_1 + b_1 - d_1(1-b_1) - c_1(1-a_1)}{2} = \frac{a_1 + b_1 - d_1 - c_1}{2} + \frac{a_1c_1 + b_1d_1}{2} = S(A_1) + \frac{a_1c_1 + b_1d_1}{2} ,$$

and

$$L(\overline{A}_1) = \frac{c_1 + d_1 - b_1(1-d_1) - a_1(1-c_1)}{2} = \frac{c_1 + d_1 - b_1 - a_1}{2} + \frac{a_1c_1 + b_1d_1}{2} = \frac{a_1c_1 + b_1d_1}{2} - S(A_1) ,$$

then

$$L(A_1) \ne L(\overline{A}_1) .$$

Note 1. The accuracy functions $M(A)$ of Ye (2009) and $L(A)$ of Lakshmana Gomathi Nayagam and Sivaraman (2012) do not satisfy the Symmetry property. Moreover, we can prove that $L(A)$ satisfy the Monotonicity and Symmetry properties of score function. Thus, $L(A)$ is a score function more than an accuracy function.

2. Nonlinear functions of IVIFNs based on the normal distribution

IVIFNs are more complicate than IFNs because that the membership and non-membership functions of the former are interval numbers. It is an established fact that interval numbers are nonlinear functions and can not be compared directly (Bortolan, Degani 1985). Therefore, the score and accuracy functions can not be the straight forward extensions of their respective proposals for the case of IFNs, which are linear functions. Considering that the numbers within the interval sometimes do not mean the same for decision makers, Ahn (2006) assumed that they are distributed by the normal distribution. Motivated by this idea, this article will propose some new score and accuracy functions for ranking IVIFNs. The novelty of these functions is that they are normally distributed.

Definition 6 *(Normal distribution)***.** Let x ($x \in [a,b]$) be the continuous random variable, and then we define its probability density function as:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-[(x-u)^2/2\sigma^2]}, \tag{5}$$

where, the mean and variance of the normal distribution can be assumed to be $u = (a+b)/2$ and $\sigma^2 = (b-a)^2/4$, respectively.

Consequently, for any interval-valued intuitionistic fuzzy number $A = \left([a,b],[c,d]\right)$, its membership function and non-membership function can be approximated by the normal distribution,where $\tilde{\mu}_A(x) \sim N\left((a+b)/2,\ (b-a)^2/4\right)$ and $\tilde{v}_A(x) \sim N\left((c+d)/2,\ (d-c)^2/4\right)$, respectively.

Definition 7 (*Normal distribution based score function*)**.** Let $A = \left([a,b],[c,d]\right)$ be an interval-valued intuitionistic fuzzy number, a score function $\tilde{S}$ of A can be represented as follows:

$$\tilde{S} = \tilde{\mu}_A(x) - \tilde{v}_A(x) \sim N\left((a+b-c-d)/2,\ [(b-a)^2+(d-c)^2]/4\right), \tag{6}$$

where the mean and variance of $\tilde{S}$ are $u_{\tilde{S}} = (a+b-c-d)/2$ and $\sigma^2_{\tilde{S}} = [(b-a)^2+(d-c)^2]/4$, respectively. The larger the value of $u_{\tilde{S}}$, the more the score degree of the interval-valued intuitionistic fuzzy value $\tilde{S}$.

Theorem 1. Let $A_i = \left([a_i,b_i],[c_i,d_i]\right)(i=1,2,\ldots,m)$ be m interval-valued intuitionistic fuzzy numbers and $\tilde{S}_i(i=1,2,\ldots,m)$ be their associated score functions. Then, the addition of these independent interval numbers $\tilde{S}_i(i=1,2,\ldots,m)$, each of which is normally distributed, is also normally distributed with a mean of $u_{\tilde{S}_i}$ and a variance of $\sigma^2_{\tilde{S}_i}$:

$$w_1\tilde{S}_1 + w_2\tilde{S}_2 + \cdots + w_m\tilde{S}_m \sim N(u,\sigma^2), \tag{7}$$

with

$$u = w_1u_{\tilde{S}_1} + w_2u_{\tilde{S}_2} + \cdots + w_mu_{\tilde{S}_m} = \sum_{i=1}^{m} w_iu_{\tilde{S}_i}; \tag{8}$$

$$\sigma^2 = w_1^2\sigma^2_{\tilde{S}_1} + w_2^2\sigma^2_{\tilde{S}_2} + \cdots + w_m^2\sigma^2_{\tilde{S}_m} = \sum_{i=1}^{m} w_i^2\sigma^2_{\tilde{S}_i}. \tag{9}$$

Proof: Since $\tilde{S}_i (i=1,2,\ldots,m)$ is normally distributed, we can obtain

$$\tilde{S}_i \sim N(u_{\tilde{S}_i}, \sigma^2_{\tilde{S}_i}),$$

then

$$w_i\tilde{S}_i \sim N(w_i u_{\tilde{S}_i}, {w_i}^2\sigma^2_{\tilde{S}_i}),$$

thus

$$w_1\tilde{S}_1 + w_2\tilde{S}_2 + \cdots + w_m\tilde{S}_m \sim N(\sum_{i=1}^{m} w_i u_{\tilde{S}_i}, \sum_{i=1}^{m} w_i^2 \sigma^2_{\tilde{S}_i}),$$

which has completed the proof of Theorem 1.

Definition 8 *(Normal distribution based accuracy function)***.** Let $A = ([a,b],[c,d])$ be an interval-valued intuitionistic fuzzy number, an accuracy function $\tilde{H}$ of A can be represented as follows:

$$\tilde{H} = \tilde{\mu}_A(x) + \tilde{\nu}_A(x) \sim N\Big((a+b+c+d)/2,\ \ [(b-a)^2 + (d-c)^2]/4\Big), \tag{10}$$

where the mean and variance of $\tilde{H}$ are $u_{\tilde{H}} = (a+b+c+d)/2$ and $\sigma^2_{\tilde{H}} = [(b-a)^2 + (d-c)^2]/4$, respectively. The larger the value of $u_{\tilde{H}}$, the more the accuracy degree of the interval-valued intuitionistic fuzzy value $\tilde{H}$.

Theorem 2. Let $A_i = ([a_i,b_i],[c_i,d_i])(i=1,2,\ldots,m)$ be m interval-valued intuitionistic fuzzy numbers and $\tilde{H}_i (i=1,2,\ldots,m)$ be their associated accuracy functions. Then, the addition of these independent interval numbers $\tilde{H}_i (i=1,2,\ldots,m)$, each of which is normally distributed, is also normally distributed with a mean of $u_{\tilde{H}_i}$ and a variance of $\sigma^2_{\tilde{H}_i}$:

$$w_1\tilde{H}_1 + w_2\tilde{H}_2 + \cdots + w_m\tilde{H}_m \sim N(\overline{u}, \overline{\sigma}^2), \tag{11}$$

with

$$\overline{u} = w_1 u_{\tilde{H}_1} + w_2 u_{\tilde{H}_2} + \cdots + w_m u_{\tilde{H}_m} = \sum_{i=1}^{m} w_i u_{\tilde{H}_i}; \tag{12}$$

$$\overline{\sigma}^2 = w_1^2\sigma^2_{\tilde{H}_1} + w_2^2\sigma^2_{\tilde{H}_2} + \cdots + w_m^2\sigma^2_{\tilde{H}_m} = \sum_{i=1}^{m} w_i^2 \sigma^2_{\tilde{H}_i}. \tag{13}$$

Proof: Since $\tilde{H}_i (i=1,2,\ldots,m)$ is normally distributed, we can obtain:

$$\tilde{H}_i \sim N(u_{\tilde{H}_i}, \sigma^2_{\tilde{H}_i}),$$

then

$$w_i\tilde{H}_i \sim N(w_i u_{\tilde{H}_i}, {w_i}^2\sigma^2_{\tilde{H}_i}),$$

thus

$$w_1\tilde{H}_1 + w_2\tilde{H}_2 + \cdots + w_m\tilde{H}_m \sim N(\sum_{i=1}^{m} w_i u_{\tilde{H}_i}, \sum_{i=1}^{m} w_i^2 \sigma^2_{\tilde{H}_i}),$$

which has completed the proof of Theorem 2.

Definition 9 *(Normal distribution based hesitation function)***.** Let $A = ([a,b],[c,d])$ be an interval-valued intuitionistic fuzzy number, a hesitation function $\tilde{\pi}$ of A can be represented as follows:

$$\tilde{\pi} = 1 - \tilde{\mu}_A(x) - \tilde{\nu}_A(x) \sim N\Big((2-a-b-c-d)/2,\ \ [(b-a)^2 + (d-c)^2]/4\Big), \tag{14}$$

where the mean and variance of $\tilde{\pi}$ are $u_{\tilde{\pi}} = (2-a-b-c-d)/2$ and $\sigma^2_{\tilde{\pi}} = [(b-a)^2 + (d-c)^2]/4$, respectively. The larger the value of $u_{\tilde{\pi}}$, the more the hesitation degree of the interval-valued intuitionistic fuzzy value $\tilde{\pi}$. The larger the value of $\sigma^2_{\tilde{\pi}}$, the smaller the hesitation degree of the interval-valued intuitionistic fuzzy value $\tilde{\pi}$.

Theorem 3. Let $A_i = ([a_i, b_i], [c_i, d_i])(i = 1, 2, \ldots, m)$ be m interval-valued intuitionistic fuzzy numbers and $\tilde{\pi}_i (i = 1, 2, \ldots, m)$ be their associated hesitation function. Then, the addition of these independent interval numbers $\tilde{\pi}_i (i = 1, 2, \ldots, m)$, each of which is normally distributed, is also normally distributed with a mean of $\bar{\bar{u}}$ and a variance of $\bar{\bar{\sigma}}_2$:

$$w_1\tilde{\pi}_1 + w_2\tilde{\pi}_2 + \cdots + w_m\tilde{\pi}_m \sim N(\bar{\bar{u}}, \bar{\bar{\sigma}}^2), \tag{15}$$

with

$$\bar{\bar{u}} = w_1 u_{\tilde{\pi}_1} + w_2 u_{\tilde{\pi}_2} + \cdots + w_m u_{\tilde{\pi}_m} = \sum_{i=1}^{m} w_i u_{\tilde{\pi}_i}; \tag{16}$$

$$\bar{\bar{\sigma}}^2 = w_1^2 \sigma^2_{\tilde{\pi}_1} + w_2^2 \sigma^2_{\tilde{\pi}_2} + \cdots + w_m^2 \sigma^2_{\tilde{\pi}_m} = \sum_{i=1}^{m} w_i^2 \sigma^2_{\tilde{\pi}_i}. \tag{17}$$

Proof: Since $\tilde{\pi}_i (i = 1, 2, \ldots, m)$ is normally distributed, we can obtain:

$$\tilde{\pi}_i \sim N(u_{\tilde{\pi}_i}, \sigma^2_{\tilde{\pi}_i}),$$

then

$$w_i\tilde{\pi}_i \sim N(w_i u_{\tilde{\pi}_i}, w_i{}^2 \sigma^2_{\tilde{\pi}_i}),$$

thus

$$w_1\tilde{\pi}_1 + w_2\tilde{\pi}_2 + \cdots + w_m\tilde{\pi}_m \sim N(\sum_{i=1}^{m} w_i u_{\tilde{\pi}_i}, \sum_{i=1}^{m} w_i^2 \sigma^2_{\tilde{\pi}_i}),$$

which has completed the proof of Theorem 3.

Based on these three types of values for IVIFNs: the score function S, the accuracy function H and hesitation function π, we shall present a method for the comparison between any two IVIFNs as follows:

Definition 10 *(Order relation of IVIFNs).* Let $\tilde{a}_1 = ([a_1, b_1], [c_1, d_1])$ and $\tilde{a}_2 = ([a_2, b_2], [c_2, d_2])$ be two interval-valued intuitionistic fuzzy values, $\tilde{S}(\tilde{a}_1) \sim N((a_1 + b_1 - c_1 - d_1)/2, [(b_1 - a_1)^2 + (d_1 - c_1)^2]/4$ and $\tilde{S}(\tilde{a}_2) \sim N((a_2 + b_2 - c_2 - d_2)/2, [(b_2 - a_2)^2 + (d_2 - c_2)^2]/4$ be their associate score functions, respectively, and let $H(\tilde{a}_1) \sim N((a_1 + b_1 + c_1 + d_1)/2, [(b_1 - a_1)^2 + (d_1 - c_1)^2]/4)$ and $H(\tilde{a}_2) \sim N((a_2 + b_2 + c_2 + d_2)/2, [(b_2 - a_2)^2 + (d_2 - c_2)^2]/4)$ be their associate accuracy functions, respectively, and let $\pi(\tilde{a}_1) \sim N((1 - a_1 - b_1 - c_1 - d_1)/2, [(b_1 - a_1)^2 + (d_1 - c_1)^2]/4)$ and $\pi(\tilde{a}_2) \sim N((1 - a_2 - b_2 - c_2 - d_2)/2, [(b_2 - a_2)^2 + (d_2 - c_2)^2]/4)$ be their associate hesitation functions, respectively, then:

(1) If $u_{S(\tilde{a}_1)} < u_{S(\tilde{a}_2)}$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$;

(2) If $u_{S(\tilde{a}_1)} = u_{S(\tilde{a}_2)}$, and, then if $u_{H(\tilde{a}_1)} < u_{H(\tilde{a}_2)}$, $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$;

(3) If $u_{S(\tilde{a}_1)} = u_{S(\tilde{a}_2)}$, and $u_{H(\tilde{a}_1)} = u_{H(\tilde{a}_2)}$, then if $\sigma^2_{\pi(\tilde{a}_1)} < \sigma^2_{\pi(\tilde{a}_2)}$, $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$;

(4) If $u_{S(\tilde{a}_1)} = u_{S(\tilde{a}_2)}$, $u_{H(\tilde{a}_1)} = u_{H(\tilde{a}_2)}$, and $\sigma^2_{\pi(\tilde{a}_1)} = \sigma^2_{\pi(\tilde{a}_2)}$, then $\tilde{a}_1$ and $\tilde{a}_2$ represent the same information, denoted $\tilde{a}_1 = \tilde{a}_2$.

In the following, we use the above approach to compare two IVIFMS $\tilde{a}_1$ and $\tilde{a}_2$ in Example 1 and Example 2, which shows that our functions are more reasonable than the functions proposed in (Xu, Chen 2007; Ye 2009; Lakshmana Gomathi Nayagam, Sivaraman 2012).

Example 3. (*Example 1 continuation*)

By Dentitions 7 and 8, we can obtain that $u_{S(\tilde{a}_1)} = u_{S(\tilde{a}_2)} = -0.15$ and $u_{\tilde{H}(\tilde{a}_1)} = u_{\tilde{H}(\tilde{a}_2)} = 0.55$, respectively. Therefore, we can not use these values to compare $\tilde{a}_1$ and $\tilde{a}_2$. However, according to Defintion 9, we calculate the variance values of hesitation inedx $\sigma^2_{\pi(\tilde{a}_1)} = 0.045$ and $\sigma^2_{\pi(\tilde{a}_2)} = 0.0125$, respectively. According to Eq. (3) in Definition 10, we can get that $\tilde{a}_1$ is better than $\tilde{a}_2$.

Note 2: All of Definitions 2, 3 and 4 fail to rank interval-valued intuitionistic fuzzy values for two alternatives in this example. However, our approach is applicable.

Example 4. (*Example 2 continuation*)

According to Dentitions 7 and 8, we get that $u_{S(\tilde{a}_1)} = u_{S(\tilde{a}_2)} = -0.2$ and $u_{H(\tilde{a}_1)} = u_{H(\tilde{a}_2)} = 0.6$. There is no difference in $\tilde{a}_1$ and $\tilde{a}_2$ based on these two functions. However, by Dentition 9, we obtain the variance value of hesitation index $\sigma^2_{\pi(\tilde{a}_1)} = 0.01$ and $\sigma^2_{\pi(\tilde{a}_2)} = 0.09$, respectively. According to Eq. (3) in Definition10, we can get that $\tilde{a}_2$ is better than $\tilde{a}_1$.

Note 3: All of Definitions 2, 3 and 4 fail to rank interval-valued intuitionistic fuzzy values for two alternatives in this example. However, our approach is still applicable.

In the above two examples, our approach can rank these interval-valued intuitionistic fuzzy sets correctly. The advantage of our approach is the use of the variance value of hesitation index, which can provide more information for interval-valued intuitionistic fuzzy numbers.

3. An optimization model for attribute weight based on nonlinear functions

In some decision making problems, due to the increasing complexity of many practical decision situations, the DM may not be confident in providing exact values for attribute weights. Instead, the decision maker (DM) may only possess partial knowledge about attribute weights. The types of Q provided by group members are linearly unequal constraints, which can be constructed by the following forms (Kim, Ahn 1999):

(1) A weak ranking: $\{w_i \geq w_j\}$;

(2) A strict ranking: $\{w_i - w_j \geq a_i\}$;

(3) A ranking with multiples: $\{w_i \geq a_i w_j\}$;

(4) An interval form: $\{a_i \leq w_i \leq a_i + \varepsilon_i\}$;

(5) A ranking of differences: $\{w_i - w_j \geq w_k - w_l\}$, for $j \neq k \neq l$,

where α_i and ε_i are nonnegative constants.

From Eq. (8), we know that the score of alternative x_i is based on the value of $S_i = \sum_{i=1}^{n} w_i \tilde{S}_{ij}$. Obviously, the greater the value S_i, the better the alternative x_i. Therefore, we can build the following three optimization models to drive attribute weight:

$$\text{Max } S_i = \sum_{i=1}^{n} w_i \tilde{S}_{ij}$$
$$s.t.\begin{cases} W \in Q \\ \sum_{i=1}^{n} w_i = 1 \\ 0 \le w_i \end{cases} \tag{18}$$

and

$$\text{Max } H_i = \sum_{i=1}^{n} w_i \tilde{H}_{ij}$$
$$s.t.\begin{cases} W \in Q \\ \sum_{i=1}^{n} w_i = 1 \\ 0 \le w_i \end{cases} \tag{19}$$

and

$$\text{Max } \sigma^2_{\tilde{S}_i} = \sum_{i=1}^{n}\sum_{j=1}^{m} w_i^2 \sigma^2_{\tilde{S}_{ij}}$$
$$s.t.\begin{cases} W \in Q \\ \sum_{i=1}^{n} w_i = 1 \\ 0 \le w_i \end{cases} . \tag{20}$$

Since all of the alternatives in the MADM problems are competitive, the above multi-objective programming models could be further aggregated into a single objective programming as follows:

$$\text{Max } S = \sum_{i=1}^{n}\sum_{j=1}^{m} w_i u_{\tilde{S}_{ij}}$$
$$s.t.\begin{cases} W \in Q \\ \sum_{i=1}^{n} w_i = 1 \\ 0 \le w_i \end{cases} \tag{21}$$

and

$$\text{Max } \tilde{H} = \sum_{i=1}^{n}\sum_{j=1}^{m} w_i u_{\tilde{H}_{ij}}$$
$$s.t.\begin{cases} W \in Q \\ \sum_{i=1}^{n} w_i = 1 \\ 0 \le w_i \end{cases} \tag{22}$$

and

$$\text{Max } \sigma^2_{\tilde{S}} = \sum_{i=1}^{n}\sum_{j=1}^{m} w_i^2 \sigma^2_{\tilde{S}}$$
$$s.t.\begin{cases} W \in Q \\ \sum_{i=1}^{n} w_i = 1 \\ 0 \le w_i \end{cases}. \quad (23)$$

Because expressions (21), (22) and (23) are there maximization problems with the same constraints, they can be combined to formulate the following optimization program:

$$\text{Max } Z = \sum_{i=1}^{n}\sum_{j=1}^{m} w_i u_{\tilde{S}_{ij}} + w_i u_{\tilde{H}_{ij}} + w_i^2 \sigma^2_{\tilde{H}_{ij}}$$
$$s.t.\begin{cases} W \in Q \\ \sum_{i=1}^{n} w_i = 1 \\ 0 \le w_i \end{cases}. \quad (24)$$

By resolving the above optimization program, we can obtain an optimal weight vector W^T.

4. Multi-criteria fuzzy decision-making method based on the new novel score function and accuracy function

In this section, we shall present an approach for Multi-criteria fuzzy decision-making problems with interval-valued intuitionistic fuzzy numbers based on the new novel score function and accuracy function.

Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of alternatives and let $C = \{C_1, C_2, \ldots, C_m\}$ be a set of criteria. Assume that the weight of the criterion $C_j (j = 1,2,\ldots,n)$, entered by the decision-maker, is $w_j, w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$. In this case, the characteristic of the alternative A_i is represented by an IVIFS:

$$A_i = \left\{ \left\langle C_j, [\mu_{iL}(C_j), \mu_{iU}(C_j)], [\nu_{iL}(C_j), \nu_{iU}(C_j)] \right\rangle | C_j \in C \right\},$$

where $0 \le \mu_{iU}(C_j) + \nu_{iU}(C_j) \le 1, \mu_{iL}(C_j) \ge 0, \nu_{iL}(C_j) \ge 0,\ j = 1,2,\ldots,n$ and $i = 1,2,\ldots,m$. The IVIFS value that is the pair of intervals $\mu_{A_i}(C_j) = [a_{ij}, b_{ij}], \nu_{A_i}(C_j) = [c_{ij}, d_{ij}]$ for $C_j \in C$ is denoted by $\alpha_{ij} = \left([a_{ij}, b_{ij}], [c_{ij}, d_{ij}]\right)$, where $[a_{ij}, b_{ij}]$ indicates the degree that the alternative A_i satisfies the criterion C_j given by the decision maker, $[c_{ij}, d_{ij}]$ indicates the degree that the alternative A_i does not satisfy the criterion C_j given by the decision maker, $[a_{ij}, b_{ij}] \in [0,1], [c_{ij}, d_{ij}] \in [0,1]$. Therefore, we can elicit a decision matrix $D = (\alpha_{ij})_{m \times n}$. The next four steps can summarize the procedure of applying this method.

Case 1. The DMs have complete weight information.

Step 1. By Eq. (5), we calculate the score matrix $S=(s_{ij})_{m\times n}$ of $D=(\alpha_{ij})_{m\times n}$.

Step 2. Calculate the score values, accuracy values and variance values of interval-valued intuitionistic fuzzy value α_{ij} ($i=1,2,\ldots,m$, $j=1,2,\ldots,n$) by using Eq. (5) and Eq. (11).

Step 3. Aggregate the score values, accuracy values and variance values by using Eq. (6) and Eq. (10), respectively.

Step 4. Rank the alternative $A=\{A_1,A_2,\ldots,A_m\}$ and select the best one(s) according to Definition 9.

Step 5. End.

Case 2. The DMs have partial weight information.

Step 1. By resolve the Expression (24), we obtain an optimal weight vector W^T.

Step 2. By Eq. (5), we calculate the score matrix $S=(s_{ij})_{m\times n}$ of $D=(\alpha_{ij})_{m\times n}$.

Step 3. Calculate the score values, accuracy values and variance values of interval-valued intuitionistic fuzzy value α_{ij} ($i=1,2,\ldots,mi=1,2,\ldots,m$, $j=1,2,\ldots,n$) by using Eq. (5) and Eq. (11).

Step 4. Aggregate the score values, accuracy values and variance values by using Eq. (6) and Eq. (10), respectively.

Step 5. Rank the alternative $A=\{A_1,A_2,\ldots,A_m\}$ and select the best one(s) according to Definition 9.

Step 6. End.

5. Illustrative example

With increasing governmental regulation and stronger public awareness in environmental protection, environmental performance evaluation has become an important issue in green production. An electronic company is desirable to select its green suppliers. After pre-evaluation, four suppliers. $A_i(i=1,2,3,4)A_i(i=1,2,3,4)$ are remained as alternatives for further evaluation. Four criteria are considered as: C_1: Remanufacturing activity; C_2: Energy consumption; C_3: Hazardous waste management; C_4: Environmental certification. Since most of these criteria are qualitative, there exist some fuzziness and uncertainty in this type of decision making problem. Therefore, the assessments by interval-valued intuitionistic fuzzy numbers to four alternatives are shown in Table 1.

Table 1. Assessments of four green suppliers based on each criterion

Suppliers	Criteria			
	C_1	C_2	C_3	C_4
A_1	([0.40, 0.65], [0.10, 0.30])	([0.30, 0.50], [0.10, 0.20])	([0.50, 0.60], [0.20, 0.40])	([0.60, 0.70], [0.20, 0.30])
A_2	([0.50, 0.75], [0.10, 0.20])	([0.50, 0.60], [0.20, 0.30])	([0.45, 0.55], [0.35, 0.45])	([0.20, 0.30], [0.10, 0.20])
A_3	([0.30, 0.70], [0.10, 0.20])	([0.40, 0.70], [0.10, 0.20])	([0.40, 0.60], [0.15, 0.40])	([0.50, 0.60], [0.30, 0.40])
A_4	([0.40, 0.70], [0.30, 0.30])	([0.60, 0.80], [0.10, 0.20])	([0.30, 0.50], [0.10, 0.20])	([0.20, 0.50], [0.10, 0.40])

From Table 1, we can get the following decision making matrix:

$$D = \begin{bmatrix} ([0.40, 0.65],[0.1, 0.3]) & ([0.3, 0.5],[0.1,0.2]) & ([0.50, 0.60],[0.20,0 .40]) & ([0.6, 0.7],[0.2, 0.3]) \\ ([0.50, 0.75],[0.1, 0.2]) & ([0.5, 0.6],[0.2, 0.3]) & ([0.45, 0.55],[0.35, 0.45]) & ([0.2, 0.3],[0.1, 0.2]) \\ ([0.30, 0.70],[0.1, 0.2]) & ([0.4, 0.7],[0.1, 0.2]) & ([0.40, 0.60],[0.15, 0.40]) & ([0.5, 0.6],[0.3, 0.4]) \\ ([0.40, 0.70],[0.3, 0.3]) & ([0.6, 0.8],[0.1, 0.2]) & ([0.30, 0.50],[0.10, 0.20]) & ([0.2, 0.5],[0.1, 0.4]) \end{bmatrix}.$$

5.1. The DMs have complete weight information

Assume that the weights of C_1, C_2, C_3 and C_4 are 0.4, 0.3, 0.1 and 0.2, respectively. Then, we utilize our approach to get the most desirable alternative(s).

Step 1. According to expression (1), we calculate the associated score values of s_{ij} ($i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$) in Table 2.

Table 2. The score values of s_{ij}

Score values	Criteria			
	C_1	C_2	C_3	C_4
A_1	0.325	0.250	0.250	0.400
A_2	0.475	0.300	0.100	0.100
A_3	0.250	0.400	0.225	0.200
A_4	0.250	0.550	0.250	0.100

Using the weights vector of criteria $W = (0.4,\ 0.3,\ 0.1,\ 0.2)^T$, we obtain the overall score values S_i $(i = 1,2,3,4)$ of the alternative A_i as follows:

$$S_1 = 0.3100,\ S_2 = 0.3100,\ S_3 = 0.3225,\ S_4 = 0.3100.$$

Then, we have that $A_3 \succ A_4 = A_1 = A_2$. Obviously, the score function of Xu and Chen (2007) is not able to rank the alternatives A_1, A_2 and A_4. In the following, we further rank these there alternatives. To do that, we compute the accuracy values s_{ij} ($i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$) in Table 3.

Table 3. The accuracy values of s_{ij}

Accuracy values	Criteria			
	C_1	C_2	C_3	C_4
A_1	0.725	0.550	0.850	0.900
A_2	0.775	0.800	0.900	0.400
A_3	0.650	0.700	0.775	0.900
A_4	0.850	0.850	0.550	0.600

By applying the weights vector of criteria $W = (0.4,\ 0.3,\ 0.1,\ 0.2)^T$, we calculate the overall accuracy values H_i $(i = 1,2,3,4)$ of the alternative A_i as follows:

$$H_1 = 0.7200,\ H_2 = 0.7200,\ H_3 = 0.7275,\ H_4 = 0.7700.$$

According to Definition 10, we obtain that $A_3 \succ A_4 \succ A_1 = A_2$. However, the accuracy function of Xu and Chen (2007) is still not able to rank the alternatives A_1 and A_2. In the following, we compute the variance values of s_{ij} ($i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$) in Table 4.

Table 4. The variance values of s_{ij}

Variance values	Criteria			
	C_1	C_2	C_3	C_4
A_1	0.0256	0.0125	0.0125	0.0050
A_2	0.0181	0.0050	0.0050	0.0050
A_3	0.0425	0.0250	0.0256	0.0050
A_4	0.0225	0.0125	0.0125	0.0450

By expression (17), we calculate the overall variance values $\sigma^2_{S_i}$ $(i = 1,2,3,4)$ of the alternative A_i as follows:

$$\sigma^2_{S_1} = 0.0055,\ \sigma^2_{S_2} = 0.0036,\ \sigma^2_{S_3} = 0.0095,\ \sigma^2_{S_4} = 0.0066.$$

Then, we have $A_1 \succ A_2$. According to Definition 10, we have the ranking order of the alternatives: $A_3 \succ A_4 \succ A_1 \succ A_2$. Consequently, the variance value $\sigma^2_{S_i}$ is a useful tool when the score values and accuracy functions do not allow the proper discrimination between different interval-valued intuitionistic fuzzy numbers.

To further study the properties of the score functions, accuracy functions, and variance functions, we compute the accuracy values of Ye (2009) in Table 5.

Table 5. Ye's accuracy values of s_{ij}

M_{ij}	Criteria			
	C_1	C_2	C_3	C_4
A_1	0.250	–0.050	0.400	0.550
A_2	0.400	0.350	0.400	–0.350
A_3	0.150	0.250	0.275	0.450
A_4	0.400	0.550	–0.050	–0.050

By expression (3), we obtain Ye's overall accuracy values M_i $(i = 1,2,3,4)$ of the alternative A_i as follows:

$$M_1 = 0.2350,\ M_2 = 0.2350,\ M_3 = 0.2525,\ M_4 = 0.3100.$$

Then we obtain that $A_4 \succ A_3 \succ A_1 = A_2$, where A_1 and A_2 are not discriminated.

We also can calculate the accuracy function values of Lakshmana Gomathi Nayagam and Sivaraman (2012) in Table 6.

Table 6. Nayagam's accuracy values of s_{ij}

LA_{ij}	Criteria			
	C_1	C_2	C_3	C_4
A_1	0.885	0.630	0.840	1.130
A_2	1.150	0.880	0.605	0.280
A_3	0.870	0.980	0.750	0.790
A_4	0.830	1.320	0.630	0.420

By Definition 5, we obtain Nayagam's overall accuracy values LA_i $(i=1,2,3,4)$ of the alternative A_i as follows:

$$LA_1 = 0.8530,\ LA_2 = 0.8405,\ LA_3 = 0.8750,\ LA_4 = 0.8750,$$

Then, we obtain that $A_3 = A_4 \succ A_1 \succ A_2$. This ranking order is similar as the one by our proposed approach, which demonstrates LA is a score function more than an accuracy function. However, A_3 and A_4 are still not discriminated in this case.

5.2. The DMs have partial weight information

Step 1. The information about the attribute weights is partly known as follows:

$$Q=\left\{0.1 \le w_1 \le 0.3, 0.2 \le w_2 \le 0.4, 0.15 \le w_3 \le 0.3, 0.28 \le w_4 \le 0.4, \sum_{j=1}^{n} w_n = 1, 0 \le w_i \le 1\right\}.$$

Based on the expression (24), we build the following optimization model:

$$\text{Max } Z=\sum_{i=1}^{n}\sum_{j=1}^{m} 4.4w_1+4.4w_2+3.9w_3+3.6w_4+ 0.1w_1^2+0.06w_2^2+ 0.055w_3^2+0.06w_4^2$$

$$\begin{cases} 0.1 \le w_1 \le 0.3 \\ 0.2 \le w_2 \le 0.4 \\ 0.15 \le w_3 \le 0.3 \\ 0.28 \le w_4 \le 0.4 \\ w_1+w_2+w_3+w_4=1 \\ 0 \le w_i \le 1 \end{cases}.$$

By resolving this model, we obtain the weight vector.

Step 2. By using Eq. (7) and Eq. (8), we can obtain weighed values of each alternative as follows:

Step 3. According to Definition 10, we rank all alternatives: . And then, the most desirable alternative is.

Step 4. End.

5.3. Analysis of the nonlinear functions

The ranking method for interval-valued intuitionistic fuzzy numbers proposed in this paper has the following main advantages with respect to other methods proposed in the literature:

1. It builds some judgment criterions to study the rationality for score and accuracy functions, It is worth mentioning that this issue that has not been successfully addressed.
2. It studies some desirable properties of score and accuracy functions: Monotonicity and Symmetry.
3. It supports the decision making progress in which the information weights about attribute is partly known, i.e. it presents a nonlinearly optimized model to obtain the weights of attributes.

Finally, the ranking method proposed in this paper differs with respect to the existing models (Xu, Chen 2007; Ye 2009; Lakshmana Gomathi Nayagam, Sivaraman 2012) in the following aspects:

1. It allows the presence of the nonlinear score and accuracy functions, which are based on the normal distribution.
2. It ranks interval-valued intuitionistic fuzzy numbers by incorporating score function, accuracy function, and variance function. Therefore, it has more information than Xu and Chen's score function and Ye's accuracy function.
3. It proves thatof Lakshmana Gomathi Nayagam and Sivaraman (2012) is a score function more than an accuracy function.

Conclusions

This article develops the nonlinear score, accuracy and hestitation functions of IVIFNs based on the normal distribution. Then, we study their desirable properties: Monotonicity and Symmetry. Based on these nonlinear functions, an approach for ranking interval-valued intuitionistic fuzzy numbers is proposed. The novelty of this approach is that it contains three values: the score values, variance values and accuracy values. As a result, it can give more information than Xu and Chen's score function and Ye's accuracy function. By combining these nonlinear functions, we investigate a multi-criterion decision-making method with IVIFNs in which two cases are considered: the attribute weight information is known and particularly known. Finally, an illustrative example is provided to illustrate our proposed approach. Considered that, sometimes, the interval numbers may follow other distribution, such as distribution. In our future work, we shall focus on the new functions of interval-valued intuitionistic fuzzy numbers based on other distributions.

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